Geometric Simplicity Theory

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Definition

Let S be a set. If an operation $cl : \mathcal{P}(S) \to \mathcal{P}(S)$ satisfies the following properties, then we say that (S, cl) is a *geometry*.

1. For $A \subseteq S$, $A \subseteq cl(A) = cl(cl(A))$.
2. For $A \subseteq B \subseteq S$, $cl(A) \subseteq cl(B)$.
3. For $A \subseteq S$ and $a, b \in S$, if $a \in cl(Ab) \setminus cl(A)$, then $b \in cl(Aa)$.
4. If $a \in cl(A)$, then $a \in cl(A_0)$, for some finite $A_0 \subseteq A$.

We say two geometries (S_0, cl_0) and (S_1, cl_1) are *equivalent* if there is a bijection $f : S_0' / \sim_0 \to S_1' / \sim_1$ preserving closure relations, where for $a, b \in S_i' := S_i - cl(\emptyset)$, $a \sim_i b$ iff $cl_i(a) = cl_i(b)$.

Let (S, cl) be a geometry.

We say that $A (\subseteq S)$ is *independent* if $a \notin cl((A \setminus \{a\})$ for all $a \in A$.

Given B, a subset $B_0 \subseteq B$ is said to be a *basis* for B if $B \subseteq cl(B_0)$ and B_0 is independent.

It follows that any two bases for B have the same cardinality, denoted by $dim(B)$.

Any $A \subseteq S$ gives a *localized* geometry on S defined by $cl_A(B) = cl(A \cup B)$, and a notion of dimension *over* A ($dim(-/A)$).

For $A, B, C \subseteq S$, if $dim(A'/C) = dim(A'/B \cup C)$ for any finite $A' \subseteq A$, then we say that A is *independent from* B *over* C.

Definition

Let (S, cl) be a geometry.

1. (S, cl) is *trivial* if $cl(A) = \bigcup\{cl(\{a\}) : a \in A\}$ for all $A \subseteq S$.
2. (S, cl) is *modular* if X is independent from Y over $X \cap Y$ for all closed X, Y, or equivalently, if $dim(X) + dim(Y) = dim(X \cup Y) + dim(X \cap Y)$ for finite dimensional closed X and Y.
3. (S, cl) is *locally modular* if it is modular over some point in S.
4. (S, cl) is *locally finite* if the closure of a finite set is finite.

Let T be strongly minimal (so stable), i.e. any definable subset of $\mathcal{M}^1$ is either finite or co-finite. Then $(\mathcal{M}, \text{acl})$ forms a geometry. (Recall $\text{acl}(A) := \{c \in \mathcal{M}^1 |$ $\text{tp}(a/A)$ is algebraic, (i.e. has only finitely many solutions). $\}$

Example

1. Infinite set: Trivial geometry.
2. Vector space $(V, +, r)_{r \in F}$: Modular geometry.
   Affine space $(V, \lambda_r, G)_{r \in F}$ where $\lambda_r(u, v) = ru + (1 - r)v$, $G(u, v, w) = u - v + w$: Locally modular geometry
3. Complex field $\mathbb{C} = (\mathbb{C}, +, -, \times, 0, 1)$: Non-modular geometry.

- Zilber's Principle (early 80s) roughly says that above are more or less all the examples of strongly minimal structures. Namely, any strongly minimal structure is either trivial, locally modular, or interpreting an infinite field (which must be acl-field by Macintyre's result).
- Hrushovski (early 90s) constructed counterexamples of Zilber's Principle using his ingenious construction method, which are not locally modular where no infinite group is interpretable. But later on, he and Zilber suggested the so-called 'Zariski condition', and *proved* Zilber's Principle under the constraint. It is well-known that Hrushovski (mid 90s) solved function field version Mordell-Lang conjecture in number theory, by the spectacular applications of the mentioned results.
- Zilber's Principle also makes sense in the o-minimal theory context, and Starchenko and Peterzil solved it fully and positively.

If $\mathcal{M}^1$ is strongly minimal, then there is a unique complete 1-type p over $\emptyset$ having SU-rank 1. p indeed is a strong type. If D is the solution set of the type, then (D, cl) where $cl(-) = \text{acl}(-) \cap D$ has the equivalent geometry to $(\mathcal{M}^1, \text{acl})$. Hence it suffices to pay our attention to (D, cl). There two notions of independence (inside D) coincide and $\dim(-/A) = SU(-/A)$.

Fix such $D = (D, cl)$ in a simple theory.

Question

1. Relationship between D being 1-based and (locally) modular?
2. If D is locally modular, then can we recover a vector space from D?
3. If D is not modular, then when we get a field and what kind of a field we get?

Definition

D is *linear* if for any $a, b \in D^1$, and $A \subseteq D$, if $SU(ab/A) = 1$, then $SU(e) \leq 1$ where $e = \mathrm{Cb}(ab/A)$.

Fact

(Buechler, Hrushovski) D stable. TFAE.

- (D, cl) *is locally modular (i.e. for* $A, B \subseteq D$ *and* $d \in D^1$, $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(Ad) \cap \mathrm{acl}(Bd) \cap D} B$*).*
- (D, cl) *is linear.*
- (D, cl_A) *is modular for some* A*.*
- D *is 1-based (i.e. for* $A, B \subseteq D$, $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(A) \cap \mathrm{acl}(B)} B$*).*

The fact no longer holds in general if D is simple.
$(V, +, r; P)$ where P is unary generic predicate in V. Then $D =$ (solution set of)$P(x)$ is 1-based but not locally modular. We have to get missing points.

Definition

$G(D)$ denotes the collection of all SU-rank 1 elements in $D^{eq} := \mathrm{dcl}(D)$.

$D \subseteq G(D)$ and $(G(D), cl)$ also forms a geometry where again $cl(-) = \mathrm{acl}(-) \cap G(D)$.

Fact

(E. Vassiliev; de Piro, K) D simple. TFAE.

- *D is 1-based.*
- *D is linear.*
- *$G(D)$ has a modular geometry.*
- *$(G(D), cl_A)$ is modular, for any (some) small A.*

Fact

(Zilber) D stable and ω-categorical.

- *Then D is 1-based.*
- *If D is non-trivial, then a vector space over a finite field having the same geometry can definably be recovered.*

Hrushovski constructed non 1-based ω-categorical simple D using a variation of his construction method mentioned before.

Fact

(Doyen and Hubaut) D stable.

- *If D has non-trivial modular geometry, then it is equivalent to that of some projective space over a division ring.*
- *If ω-categorical D has non-modular but locally modular geometry, then it is equivalent to that of some Affine space over a finite field.*

Fact

(dePiro, K) D simple.

- *If D has non-trivial modular geometry, then again it is a projective geometry over some division ring. More generally, if D is nontrivial 1-based, then $G(D)$ has a projective geometry over a division ring.*
- *The second ω-categorical case is not generalized to the simple context.*

Even if simple D is ω-categorical, $G(D)$ need not be definable. But it is essentially so when D is 1-based.

Fact

(dePiro, K) D 1-based. If $u \in G(D)$, then $u \in \mathrm{acl}(u_0 u_1 u_2)$ for some $u_i \in D$.
More precisely if (x, y) is a fixed independent pair from D. Then $u \in acl(x'y'z)$ where $x', y', z \in D$ and $tp(xy) = tp(x'y')$.

Corollary

If D is ω-categorical and non-trivial 1-based, then D has a strongly minimal reduct, and a vector space over a finite field can definably be recovered.

Proof of Fact: Suppose that $u \in \mathrm{dcl}(a_1 ... a_{n+1})$ $(a_i \in D)$ with the induction hypothesis for n. We will verify Fact for $n+1$. We can assume that $\{a_1, ..., a_{n+1}\}$ is independent. Now let $g = \mathrm{Cb}(ua_1/a_2 ... a_{n+1})$. Thus $SU(g) = 1$ and $g \in G(D)$. Since $a_1 \notin \mathrm{acl}(g) \subseteq \mathrm{acl}(a_2 ... a_{n+1})$, u is in the line generated by $\{g, a_1\}$ (*). Then, since $g \in \mathrm{acl}(a_2 ... a_{n+1})$, by the induction hypothesis, $g \in acl(bcd)$ for some $c, d \in D$ such that $tp(cd) = tp(xy)$. If g is already in $acl(cd)$, then from (*), $u \in acl(ga_1) \subseteq acl(cda_1)$. Thus, in this case, Fact holds.

Therefore we only need to consider the case when $g \notin \mathrm{acl}(cd)$. We can also clearly assume that $g \notin \mathrm{acl}(b)$ (otherwise we are done). Then, since $G(D)$ is modular, we can find $v \in \mathrm{acl}(cd) \cap \mathrm{acl}(bg) \cap G(D)$ such that $g \in \mathrm{acl}(bv)$ (†), and $dim(vg) = 2$. Also, at least one of $\{v, c\}$ or $\{v, d\}$ (say $\{v, d\}$) is independent (**). Now, as $d, a_1 \in D$, $\mathrm{Ltp}(d) = \mathrm{Ltp}(a_1)$. Hence, we can find v' such that $\mathrm{Ltp}(vd) = \mathrm{Ltp}(v'a_1)$ $(\star)$. In particular, $\mathrm{Ltp}(v) = \mathrm{Ltp}(v')$. Then by (*),(**),$(\star)$, we can amalgamate types $\mathrm{tp}(v'/a_1)$ and $\mathrm{tp}(v/g)$, so that we obtain $v'' \models \mathrm{tp}(v'/a_1) \cup \mathrm{tp}(v/g)$.

Hence, there are b', c' such that $\mathrm{tp}(vgb) = \mathrm{tp}(v''gb')$, $\mathrm{tp}(vdc) = \mathrm{tp}(v''a_1c')$ (by $(\star)$), and so by $(\dagger)$, $\mathrm{tp}(c'a_1) = \mathrm{tp}(xy)$, $v'' \in \mathrm{acl}(c'a_1)$ and $g \in \mathrm{acl}(b'v'') \quad (\ddagger)$. Then by (*),$(\ddagger)$,

$$u \in \mathrm{acl}(ga_1) \subseteq \mathrm{acl}(ga_1v'') \subseteq \mathrm{acl}(a_1b'v'') \subseteq \mathrm{acl}(c'a_1b').$$

Since $\mathrm{tp}(c'a_1) = \mathrm{tp}(xy)$, the $(n+1)$th induction hypothesis for Fact is deduced.

Definition

T is *trivial* if for a, b, c, A, whenever $\{a, b, c\}$ is pairwise independent over A, then $\{a, b, c\}$ is independent over A.

Fact

Suppose that T is 1-based. Then T is trivial if and only if all SU-rank 1 types are trivial.

Corollary

Let T be non-trivial, 1-based and ω-categorical. Then an infinite dimensional vector space over a finite field, in particular the infinite additive group, is definable.

Fact

Let T be supersimple and ω-categorical. Then T is 1-based if and only if T has finite SU-rank and SU-rank 1 types are 1-based.

Fact

(Cherlin, Harrington, Lachlan) If T is ω-categorical superstable, then it has finite SU-rank. Hence due to Zilber's and above results, T is 1-based.

Open Problem

If T is ω-categorical supersimple, then does it have finite SU-rank?

Definition

T is said to be *CM-trivial* if for acl-sets A, B, C, whenever $\mathrm{acl}(A \cup C) \cap \mathrm{acl}(A \cup B) = A$ then $\mathrm{Cb}(C/A) \subseteq \mathrm{acl}(\mathrm{Cb}(C/A \cup B))$.

1-based theory is CM-trivial. Hrushovski's example of non 1-based ω-categorical supersimple theory is CM-trivial.

Theorem

(Evans, Wagner)

- *If a group G is ω-categorical supersimple, then it is (finite-by-abelian)-by-finite (i.e. $(G^0)'$ is finite), and it has finite SU-rank.*
- *If T is ω-categorical supersimple and CM-trivial, then it has finite SU-rank.*

Open Problem

If T is ω-categorical supersimple, then is it CM-trivial?