Geometric Simplicity Theory

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ALC 10

Dept. Math. Yonsei University
September 1-6, 2008
Outline

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Geometric Simplicity Theory

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$T$ will be a complete theory in $\mathcal{L}$ having only infinite models. We work in a big saturated model $\mathcal{M}$ of size $\bar{\kappa}$ of $T$. $\mathcal{M}, \mathcal{N}, \ldots$ will be (small) elementary submodels of $\mathcal{M}$; $A, B, \ldots$ will be subsets of $\mathcal{M}$; $a, b, c, \ldots$ denote tuples (possibly infinite) from $\mathcal{M}$, and $c \in A$ means $c$ is a tuple from $A$; $AB$, or $Ac$ with $c = (c_1 \ldots c_n)$ often abbreviates $A \cup B$, or $A \cup \{c_1, \ldots, c_n\}$ respectively; $a \equiv_A b$ means $\text{tp}(a/A) = \text{tp}(b/A)$.

Types $p(x), q(x), \ldots$ are partial types with parameter, so are formulas $\varphi(x), \psi(x), \ldots$. For $\varphi(x, a)$, $a$ is the parameter on $\varphi(x, y) \in \mathcal{L}$. 

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Geometric Simplicity Theory
Definition

1. A formula $\varphi(x, a_0)$ *k-divides over A* if there is an $A$-indiscernible sequence $\langle a_i \mid i < \omega \rangle$ such that $\{ \varphi(x, a_i) \mid i < \omega \}$ is $k$-inconsistent.
   
   A formula $\varphi(x, a)$ *divides over A* if it $k$-divides for some positive $k$.
   
   A type $p(x)$ divides over $A$ if $p \vdash \varphi(x, a)$ and $\varphi(x, a)$ divides over $A$.

2. A type $p(x)$ *forks over A* if $p \vdash \varphi_0(x, a_0) \lor \ldots \lor \varphi_k(x, a_k)$ and $\varphi_i(x, a_i)$ divides over $A$ for each $i \leq k$. 
Write $A \downarrow_B C$ if for any $a \in A$, $\text{tp}(a/BC)$ does not fork over $B$.

(Extension) Let $A \subseteq B \subseteq C$. If $p \in S(B)$ does not fork over $A$, then there is an extension $q \in S(C)$ of $p$, which does not fork over $A$; equivalently, if $c \downarrow_A B$, then there is $c' \equiv_B c$ such that $c' \downarrow_A C$.

Fundamental Theorem of Forking

The following are equivalent.

(Symmetry) $A \downarrow_B C$ iff $C \downarrow_B A$.

(Transitivity) For $A \subseteq B \subseteq C$ and $d$, $d \downarrow_A C$ iff $d \downarrow_A B$ and $d \downarrow_B C$.

(Local Character) For any set $A$ and finite $d$, there is $A_d(\subseteq A)$ of size $\leq |\mathcal{L}|$ such that $d \downarrow_{A_d} A$. 
Definition

- $T$ is said to be **simple** if one of the equivalent properties of forking holds.
- $T$ is **unstable** if there is $\varphi(x, y)$ and $a_i, b_i \in M \ (i < \omega)$ such that $M \models \varphi(a_i, b_j) \text{ iff } i < j$.
- $T$ is **stable** if it is not unstable.

$C = \{C_0, C_1, \ldots\}$ is $A$-independent iff $C_{i+1} \downarrow_A C_0 \ldots C_i$ for all $i$ iff $C_i \downarrow_A C - \{C_i\}$ for all $i$.

Fact

- If $T$ is stable, then it is simple.
- If $T$ is simple, then forking is dividing. Hence $A \downarrow_B C$ iff for any $c \in A$, $\text{tp}(c/BC)$ does not divide over $A$. 
uncountable categorical (ACF) ⊆ superstable (DCF) ⊆ stable (SCF). Ordered field is unstable.

⊆ supersimple (PsF)

superstable ⊆ simple (PAC)

⊆ stable
Fact

(Shelah)

- If $T$ is stable, then $\downarrow$ additionally satisfies the following axiom.

(Uiqueness over a model) For $M \subseteq A$, if $c \equiv_M c'$ and $c \downarrow_M A$, $c' \downarrow_M A$, then $c \equiv_A c'$.

- The four basic axioms together with above axiom characterize stability and forking. Namely, if there is an invariant relation $\downarrow^*$ between tuple and sets satisfying the 5 axioms, then $T$ is stable and $\downarrow^* = \downarrow$.

Question

For simplicity, what axiom can substitute for (Uniqueness over a model)?
Theorem

(K, Pillay)

- If $T$ is simple then \(\vdash\) additionally satisfies the following axiom.

  (The Independence Theorem (or Type amalgamation) over a model) For $M \subseteq A_0, A_1$, if $c_0 \equiv_M c_1$, $A_0 \vdash_M A_1$, and $c_i \vdash_M A_i$ ($i = 0, 1$), then there is $c \equiv_{A_i} c_i$ such that $c \vdash_M A_0 A_1$.

- The four basic axioms together with above axiom characterize simplicity and forking.
**Definition**

We say $a, b$ have the same strong type over $A$ (write $a \equiv_A^s b$, or $stp(a/A) = stp(b/A)$) if $E(a, b)$ holds, for any finite definable equivalence relation $E(x, y)$ over $A$.

We say $a, b$ have the same Lascar (strong) type over $A$ (write $a \equiv_A^L b$, or $Ltp(a/A) = Ltp(b/A)$, or $Lstp(a/A) = Lstp(b/A)$) if $E(a, b)$ holds, for any bounded type-definable equivalence relation $E(x, y)$ over $A$. 
Theorem

- **Let** $T$ **be stable.**
  
  (*Uniqueness for strong types*) **For** $B \subseteq C$, **if** $a \equiv_B^s a'$ **and** $a \downarrow_B C$, $a' \downarrow_B C$, **then** $a \equiv_C a'$.

- **Let** $T$ **be simple.**
  
  (*Type amalgamation for Lascar types*) **For** $B \subseteq C_0, C_1$, **if** $a_0 \equiv_B^l a_1$, $C_0 \downarrow_B C_1$, **and** $a_i \downarrow_B C_i$ ($i = 0, 1$), **then** there is $a \equiv_{C_i} a_i$ **such that** $a \downarrow_B C_0 C_1$.

Open Problem

In simple $T$, does type amalgamation hold for strong types?
Definition

1. For any type $p(x)$, $D(p(x), \varphi(x, y), k)$ rank is defined by induction as follows:
   - $D(p, \varphi, k) \geq 0$ for any consistent type $p$.
   - $D(p, \varphi, k) \geq n + 1$ if for some $a$, $\varphi(x, a)$ $k$-divides over $\text{dom}(p)$, and $D(p \cup \{\varphi(x; a)\}, \varphi, k) \geq n$.

2. $D(p(x), \varphi(x, y))$ rank is defined as: $D(p, \varphi) \geq \alpha + 1$ if for some $a$, $\varphi(x, a)$ divides over $\text{dom}(p)$, and $D(p \cup \{\varphi(x; a)\}, \varphi) \geq \alpha$.

3. $D(\psi(x))$ rank is defined as: $D(\psi(x)) \geq \alpha + 1$ if there is a formula $\varphi(x)$ dividing over $\text{dom}(\psi(x))$, and $D(\psi(x) \wedge \varphi(x)) \geq \alpha$; $D(p(x)) := \min\{D(\psi_1(x) \wedge \ldots \wedge \psi_n(x)) | \psi_i(x) \in p(x)\}$. 
Fact

The following are equivalent.

- $T$ has the tree property, i.e. there exist a formula $\varphi(x, y)$, an integer $k \geq 2$ and tuples $c_\alpha$ with $\alpha \in \omega^{<\omega}$ such that
  (i) for any $\alpha \in \omega^{<\omega}$, $\{\varphi(x, c_\alpha \downarrow n) | n \in \omega\}$ is $k$-inconsistent,
  (ii) for all $\beta \in \omega^\omega$, $\{\varphi(x, c_\beta \uparrow n) | n \in \omega\}$ is consistent.
- $D(x = x, \varphi, k) = \infty$ for some $\varphi, k \geq 2$.

Fact

The following are equivalent.

- $T$ is simple.
- $D(x = x, \varphi, k) < \infty$ for any $\varphi, k$.
- $D(x = x, \varphi, k) < \omega$ for any $\varphi, k$.
- $D(x = x, \varphi, k) \leq D(x = x, \varphi) \leq D(x = x)$.
Definition

Simple $T$ is said to be low if $D(x = x, \varphi) < \omega$ for any $\varphi$.

Fact

- $T$ is low iff for any $\varphi(x, y)$, there is $k$ such that whenever $\varphi(x, a)$ divides, it $k$-divides.
- (S. Buechler) If $T$ is low, then Lascar types are strong types (i.e. $a \equiv^s_A b$ iff $a \equiv^L_A b$). Equivalently, type amalgamation holds for strong types.
Definition

- $T$ is said to be supersimple if for any finite $c$ and $A$, there is finite $A_c \subseteq A$ such that $c \upharpoonright_{A_c} A$.
- $T$ is superstable if $T$ is stable and supersimple.

Definition

For $p \in S(A)$, $SU(p) \geq \alpha + 1$ if if $p$ has a forking complete extension $q$ s.t. $SU(q) \geq \alpha$.

Fact

The following are equivalent.
- $T$ is supersimple.
- $D(x = x) < \infty$.
- $SU(p) < \infty$ for any complete $p$. 
Fact

- $T$ simple. $c \vdash_{A} B$ iff $D(tp(c/A), \varphi, k) = D(tp(c/AB), \varphi, k)$ for any $\varphi, k$.
- $T$ supersimple. $c \vdash_{A} B$ iff $SU(c/AB) = SU(c/A)$. (But $D$-rank reflects forking only for superstable $T$.)

Theorem

- (Lachlan) If $T$ is countable superstable, then either $I(T, \omega) = 1$ or $\geq \omega$.
- (K) If $T$ is countable supersimple, then either $I(T, \omega) = 1$ or $\geq \omega$. 
Stable
Stable

Simple
Stable

Superstable

Simple

Supersimple
Stable

SCF

Superstable

ACF, DCF
V.Sp

Simple

PAC

Supersimple

Psf,
V.Sp with forms
Definition

- $T$ has *elimination of imaginaries* if for any definable equivalence class $a/E$, there is finite $b \in M$ such that they are interdefinable (i.e. for any automorphism fixes the class $a/E$ setwise iff it fixes the tuple $b$).

- $T$ has *elimination of hyperimaginaries* if for any type-definable equivalence class $\bar{a}/E$ ($\bar{a}$ possibly infinite), there are definable equivalence classes $\{a_i/E_i\}_i$ such that they are interdefinable.
From now on, we assume \( T \) has elimination of imaginaries, and elimination of hyperimaginaries, (so assume \( \text{Ltp} = \text{stp} \)). **Unless mentioned otherwise \( T \) will be simple.**

**Theorem**

*(Buechler, Pillay, Wagner)* If \( T \) is supersimple, then \( T \) has elimination of hyperimaginaries.

**Fact**

If type amalgamation holds for \( \text{tp}(a/A) \), then indeed \( \text{tp}(a/A) = \text{Ltp}(a/A) = \text{tp}(a/\text{acl}(A)) \). Hence type amalgamation holds for any acl-set (e.g. a model). Such a type we call an amalgamation base.
Theorem

(Hart, K, Pillay) Any amalgamation base has a canonical base. Namely, for any amalgamation base $p = \text{tp}(c/A)$ with $A = \text{acl}(A)$, there is smallest $A_0 \subseteq A$ such that $c \downarrow_{A_0} A$ and $\text{tp}(c/A_0)$ is an amalgamation base. We write $A_0 := Cb(c/A)$.

Observation

Let $B_i = \text{acl}(B_i) \subseteq B$ ($i = 0, 1$). If $A \downarrow_{B_0} B$ and $A \downarrow_{B_1} B$, then $A \downarrow_{B_0 \cap B_1} B$: Note that $C := Cb(A/B) \subseteq B_i$, \therefore by transitivity, $A \downarrow_{C} B_0 \cap B_1$ and $A \downarrow_{B_0 \cap B_1} B$ hold.
**Definition**

Let $p(x) \in S(A)$ be an amalgamation base. We call $\varphi(x, y)$, $p$-stable if the $\varphi$-type $p \models \varphi$ has a unique nonforking $\varphi$-type extension over any $B \supseteq A$.

**Fact**

- $\varphi(x, y)$ is stable iff $\varphi(x, y)$ is $p$-stable for any amalgamation base $p(x)$.

- If $\varphi(x, y)$ is $p$-stable, then there is $\psi_\varphi(y)$ over $Cb(p)$, called the $\varphi$-definition of $p$, such that $\models \psi_\varphi(b)$ holds iff $\varphi(x, b)$ is in some nonforking extension of $p$. 
Theorem

(K) Assume $T$ is supersimple, and $p(x)$ is an amalgamation base. Then
$Cb(p) = \{ \text{names of } \psi_\varphi(y) | \varphi(x, y) \text{ is } p\text{-stable} \}.$

Definition

By a Morley sequence of $p \in S(A)$, we mean an $A$-indiscernible and independent sequence $I = \langle a_i | i < \omega \rangle$ in $p$.

$Cb(p) \subseteq I$ where $I$ is a Morley sequence of $p$. 
Consider a vector space $V = (V, +, r)_{r \in F}$. For acl-subsets $A, B$ of $V$, we have $A \downarrow_{A \cap B} B$. Equivalently, when finite dimensional, $\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$.

**Definition**

- $T$ is called *modular* if $\downarrow$ satisfies above relations between acl-sets.
- $T$ is said to be *1-based* if for any indiscernible sequence $I = \langle a_i | i < \omega \rangle$, $I - a_0$ is Morley over $a_0$. 
Observation

The following are equivalent.

(i) $T$ is 1-based.
(ii) For any $p = \text{tp}(c_0/A)$, $\text{Cb}(c_0/A) \subseteq \text{acl}(c_0)$.
(iii) $T$ is modular.

Proof: (i)$\Rightarrow$(ii) Let $e := \text{Cb}(c_0/A)$. Now there is a Morley sequence $J = \langle c_i | i \leq \omega \rangle$ of $p$. Then $c_\omega \downarrow_A I$ where $I = J - \{c_\omega\}$. Since $c_\omega \downarrow_e A| I$ and $e \subseteq I$, we have $c_\omega \downarrow_I A$. Hence $e = \text{Cb}(c_\omega/A) = \text{Cb}(c_\omega/I)$. By 1-basedness, $c_\omega \downarrow_{c_0} I$. Hence $e \in \text{acl}(c_0)$.

(ii)$\Rightarrow$(iii) Given $A, B$, we have $A \downarrow_{\text{Cb}(A/B)} B$ and $\text{Cb}(A/B) \subseteq \text{acl}(A)$. Hence by transitivity, $A \downarrow_{\text{acl}(A) \cap \text{acl}(B)} B$. 
(iii) ⇒ (i) Indiscernible \( l = \langle c_i \mid i < \omega \rangle \) is given. (iii) implies \( c_0 \upharpoonright_A c_1 c_2 \ldots \) where \( A = \text{acl}(c_0) \cap \text{acl}(c_1 c_2 \ldots) \). Now, 
\[ \text{tp}(c_1 c_2 \ldots / \text{acl}(c_0)) = \text{tp}(c_2 c_3 \ldots / \text{acl}(c_0)) \] 
Hence 
\[ A = \text{acl}(c_0) \cap \text{acl}(c_2 c_3 \ldots) \]. Again, 
\[ \text{tp}(c_0 / \text{acl}(c_2 c_3 \ldots)) = \text{tp}(c_1 / \text{acl}(c_2 c_3 \ldots)) \], and thus 
\[ A = \text{acl}(c_1) \cap \text{acl}(c_2 c_3 \ldots) \]. Then, due to (iii) again, \( c_1 \upharpoonright_A c_2 c_3 \ldots \). From transitivity, we have \( c_1 \upharpoonright_{c_0 A} c_2 c_3 \ldots \). Then \( c_1 \upharpoonright_{c_0} c_2 c_3 \ldots \) since \( A \subseteq \text{acl}(c_0) \). Thus (i) follows.