Geometric Simplicity Theory

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Outline

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T will be a complete theory in $\mathcal{L}$ having only infinite models.
We work in a big saturated model $\mathcal{M}$ of size $\bar{\kappa}$ of T.
$M, N, \ldots$ will be (small) elementary submodels of $\mathcal{M}$;
$A, B, \ldots$ will be subsets of $\mathcal{M}$;
$a, b, c, \ldots$ denote tuples (possibly infinite) from $\mathcal{M}$, and $c \in A$ means c is a tuple from A;
AB, or Ac with $c = (c_1 \ldots c_n)$ often abbreviates $A \cup B$, or $A \cup \{c_1, \ldots, c_n\}$ respectively;
$a \equiv_A b$ means $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$.
Types $p(x), q(x), \ldots$ are partial types with parameter, so are formulas $\varphi(x), \psi(x), \ldots$ For $\varphi(x, a)$, a is the parameter on $\varphi(x, y) \in \mathcal{L}$.

Definition

1.
   - A formula $\varphi(x, a_0)$ *k-divides over A* if there is an A-indiscernible sequence $\langle a_i |\ i < \omega \rangle$ such that $\{\varphi(x, a_i) | i < \omega\}$ is k-inconsistent.
   - A formula $\varphi(x, a)$ *divides over A* if it k-divides for some positive k.
   - A type $p(x)$ divides over A if $p \vdash \varphi(x, a)$ and $\varphi(x, a)$ divides over A.
2. A type $p(x)$ *forks over A* if $p \vdash \varphi_0(x, a_0) \vee \ldots \vee \varphi_k(x, a_k)$ and $\varphi_i(x, a_i)$ divides over A for each $i \leq k$.

Write $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_B C$ if for any $a \in A$, $\mathrm{tp}(a/BC)$ *does not fork over* B.

- (Extension) Let $A \subseteq B \subseteq C$. If $p \in S(B)$ does not fork over A, then there is an extension $q \in S(C)$ of p, which does not fork over A; equivalently, if $c \mathop{\smile\!\!\!\!\!\!\!\;\mid}_A B$, then there is $c' \equiv_B c$ such that $c' \mathop{\smile\!\!\!\!\!\!\!\;\mid}_A C$.

Fundamental Theorem of Forking

The following are equivalent.

- *(Symmetry)* $A \mathop{\smile\!\!\!\!\!\!\!\;\mid}_B C$ *iff* $C \mathop{\smile\!\!\!\!\!\!\!\;\mid}_B A$.
- *(Transitivity) For* $A \subseteq B \subseteq C$ *and* d, $d \mathop{\smile\!\!\!\!\!\!\!\;\mid}_A C$ *iff* $d \mathop{\smile\!\!\!\!\!\!\!\;\mid}_A B$ *and* $d \mathop{\smile\!\!\!\!\!\!\!\;\mid}_B C$.
- *(Local Character) For any set* A *and finite* d, *there is* $A_d (\subseteq A)$ *of size* $\leq |\mathcal{L}|$ *such that* $d \mathop{\smile\!\!\!\!\!\!\!\;\mid}_{A_d} A$.

Definition

- T is said to be *simple* if one of the equivalent properties of forking holds.
- T is *unstable* if there is $\varphi(x, y)$ and $a_i, b_i \in \mathcal{M}$ $(i < \omega)$ such that $\mathcal{M} \models \varphi(a_i, b_j)$ iff $i < j$.
- T is *stable* if it is not unstable.

$\mathcal{C} = \{C_0, C_1, \ldots\}$ is A-independent iff $C_{i+1} \downarrow_A C_0 \ldots C_i$ for all i iff $C_i \downarrow_A \mathcal{C} - \{C_i\}$ for all i.

Fact

- *If T is stable, then it is simple.*
- *If T is simple, then forking is dividing. Hence $A \downarrow_B C$ iff for any $c \in A$, $\mathrm{tp}(c/BC)$ does not divide over A.*

uncountable categorical (ACF) $\subseteq$ superstable (DCF) $\subseteq$ stable (SCF). Ordered field is unstable.

$$\text{superstable} \begin{array}{l} \subseteq \text{supersimple (PsF)} \\ \\ \subseteq \text{stable} \end{array} \subseteq \text{simple (PAC)}$$

Fact

(Shelah)

- *If T is stable, then $\mathop{\smile\!\!\!\!\!\!|}$ additionally satisfies the following axiom.*
  (Uniqueness over a model) For $M \subseteq A$, if $c \equiv_M c'$ and $c \mathop{\smile\!\!\!\!\!\!|}_M A$, $c' \mathop{\smile\!\!\!\!\!\!|}_M A$, then $c \equiv_A c'$.
- *The four basic axioms together with above axiom characterize stability and forking. Namely, if there is an invariant relation $\mathop{\smile\!\!\!\!\!\!|}^*$ between tuple and sets satisfying the 5 axioms, then T is stable and $\mathop{\smile\!\!\!\!\!\!|}^* = \mathop{\smile\!\!\!\!\!\!|}$.*

Question

For simplicity, what axiom can substitute for (Uniqueness over a model) ?

Theorem

(K, Pillay)

- *If T is simple then $\mathop{\smile\!\!\!\!\!\!|}$ additionally satisfies the following axiom.*
  (The Independence Theorem (or Type amalgamation) over a model) For $M \subseteq A_0, A_1$, if $c_0 \equiv_M c_1$, $A_0 \mathop{\smile\!\!\!\!\!\!|}_M A_1$, and $c_i \mathop{\smile\!\!\!\!\!\!|}_M A_i$ $(i = 0, 1)$, then there is $c \equiv_{A_i} c_i$ such that $c \mathop{\smile\!\!\!\!\!\!|}_M A_0 A_1$.
- *The four basic axioms together with above axiom characterize simplicity and forking.*

Definition

- We say a, b have the same strong type over A (write $a \equiv^s_A b$, or $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$) if $E(a, b)$ holds, for any finite definable equivalence relation $E(x, y)$ over A.
- We say a, b have the same Lascar (strong) type over A (write $a \equiv^L_A b$, or $\mathrm{Ltp}(a/A) = \mathrm{Ltp}(b/A)$, or $Lstp(a/A) = Lstp(b/A)$) if $\mathbf{E}(a, b)$ holds, for any bounded type-definable equivalence relation $\mathbf{E}(x, y)$ over A.

Theorem

- *Let T be stable.*
  (Uniqueness for strong types) For $B \subseteq C$, if $a \equiv^s_B a'$ and $a \downarrow_B C$, $a' \downarrow_B C$, then $a \equiv_C a'$.
- *Let T be simple.*
  (Type amalgamation for Lascar types) For $B \subseteq C_0, C_1$, if $a_0 \equiv^L_B a_1$, $C_0 \downarrow_B C_1$, and $a_i \downarrow_B C_i$ $(i = 0, 1)$, then there is $a \equiv_{C_i} a_i$ such that $a \downarrow_B C_0 C_1$.

Open Problem

In simple T, does type amalgamation hold for strong types ?

Definition

1. For any type $p(x)$, $D(p(x), \varphi(x,y), k)$ rank is defined by induction as follows:
   - $D(p, \varphi, k) \geq 0$ for any consistent type p.
   - $D(p, \varphi, k) \geq n+1$ if for some a, $\varphi(x,a)$ k-divides over $\mathrm{dom}(p)$, and $D(p \cup \{\varphi(x;a)\}, \varphi, k) \geq n$.
2. $D(p(x), \varphi(x,y))$ rank is defined as: $D(p, \varphi) \geq \alpha + 1$ if for some a, $\varphi(x,a)$ divides over $\mathrm{dom}(p)$, and $D(p \cup \{\varphi(x;a)\}, \varphi) \geq \alpha$.
3. $D(\psi(x))$ rank is defined as: $D(\psi(x)) \geq \alpha + 1$ if there is a formula $\varphi(x)$ dividing over $\mathrm{dom}(\psi(x))$, and $D(\psi(x) \wedge \varphi(x)) \geq \alpha$;
   $D(p(x)) := \min\{D(\psi_1(x) \wedge ... \wedge \psi_n(x)) | \ \psi_i(x) \in p(x)\}$.

Fact

The following are equivalent.

- T *has the* tree property, *i.e. there exist a formula* $\varphi(x, y)$, *an integer* $k \geq 2$ *and tuples* c_α *with* $\alpha \in \omega^{<\omega}$ *such that*
  (i) for any $\alpha \in \omega^{<\omega}$, $\{\varphi(x, c_{\alpha\frown n}) | n \in \omega\}$ *is* k*-inconsistent,*
  (ii) for all $\beta \in \omega^\omega$, $\{\varphi(x, c_{\beta\lceil n}) | n \in \omega\}$ *is consistent.*
- $D(x = x, \varphi, k) = \infty$ *for some* $\varphi, k \geq 2$.

Fact

- *The following are equivalent.*
  - T *is simple.*
  - $D(x = x, \varphi, k) < \infty$ *for any* φ, k.
  - $D(x = x, \varphi, k) < \omega$ *for any* φ, k.
- $D(x = x, \varphi, k) \leq D(x = x, \varphi) \leq D(x = x)$.

Definition

Simple T is said to be *low* if $D(x = x, \varphi) < \omega$ for any φ.

Fact

- *T is low iff for any $\varphi(x, y)$, there is k such that whenever $\varphi(x, a)$ divides, it k-divides.*
- *(S. Buechler) If T is low, then Lascar types are strong types (i.e. $a \equiv^s_A b$ iff $a \equiv^L_A b$). Equivalently, type amalgamation holds for strong types.*

Definition

- T is said to be *supersimple* if for any finite c and A, there is *finite* $A_c \subseteq A$ such that $c \mathop{\smile\!\!\!\!\!\!|}_{A_c} A$.
- T is *superstable* if T is stable and supersimple.

Definition

For $p \in S(A)$, $SU(p) \geq \alpha + 1$ if if p has a forking complete extension q s.t. $SU(q) \geq \alpha$.

Fact

The following are equivalent.

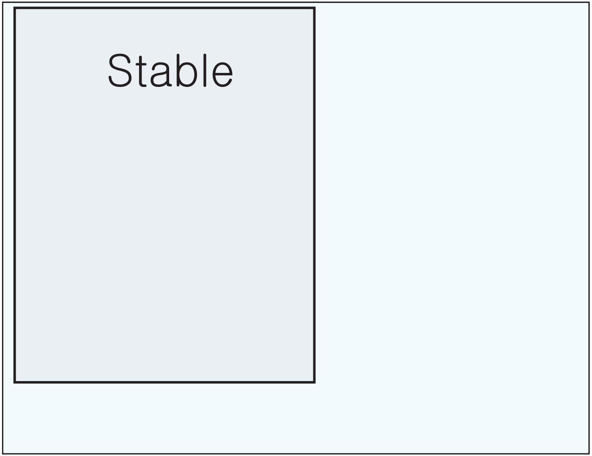
- *T is supersimple.*
- $D(x = x) < \infty$.
- $SU(p) < \infty$ *for any complete p.*

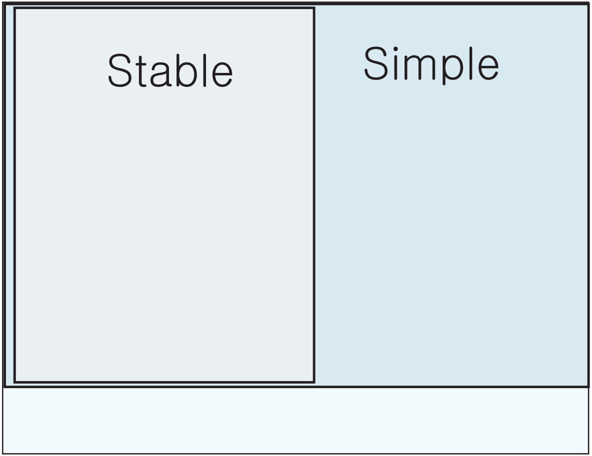
Fact

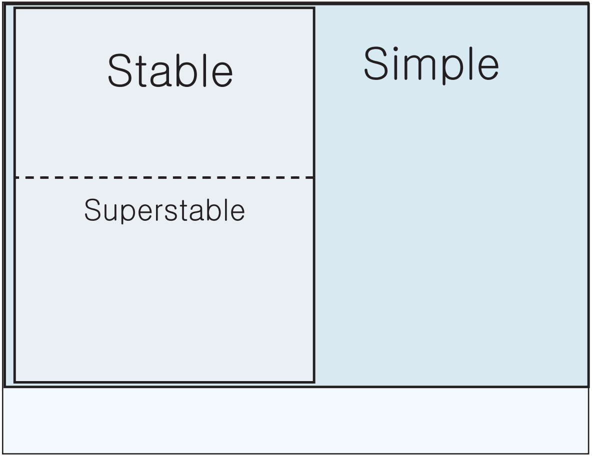
- *T simple.* $c \downarrow_A B$ *iff* $D(\text{tp}(c/A), \varphi, k) = D(\text{tp}(c/AB), \varphi, k)$ *for any* φ, k.
- *T supersimple.* $c \downarrow_A B$ *iff* $SU(c/AB) = SU(c/A)$. *(But D-rank reflects forking only for superstable* T.*)*

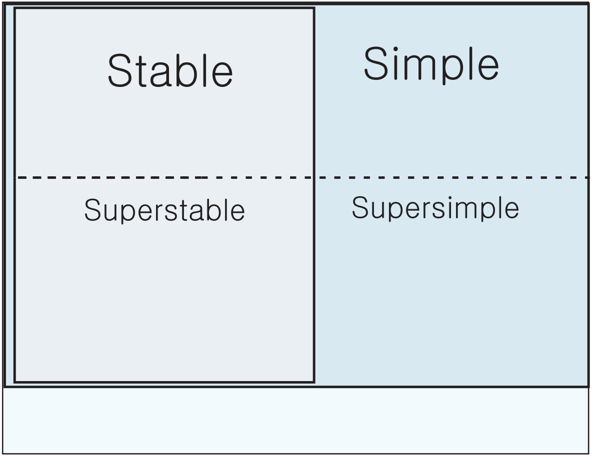
Theorem

- *(Lachlan) If* T *is countable superstable, then either* $I(T, \omega) = 1$ *or* $\geq \omega$.
- *(K) If* T *is countable supersimple, then either* $I(T, \omega) = 1$ *or* $\geq \omega$.


Stable


Stable
Simple


Stable
Superstable
Simple


Stable
Simple
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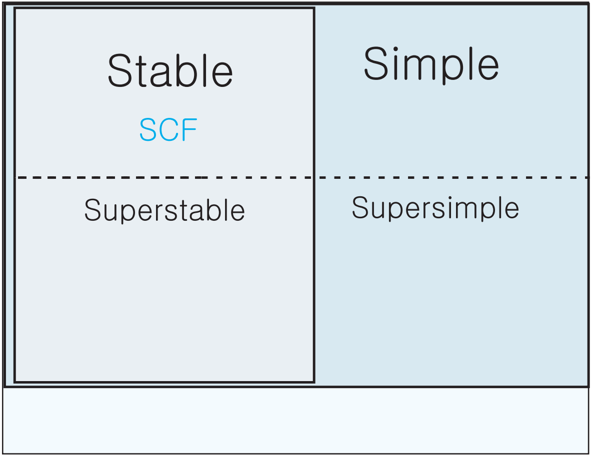
Supersimple


Stable
SCF
Simple
Superstable
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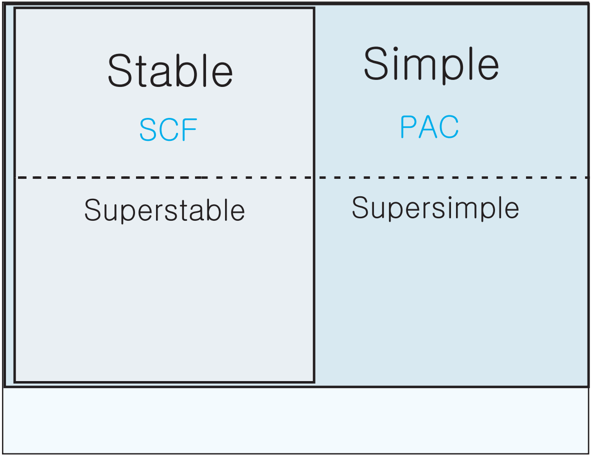
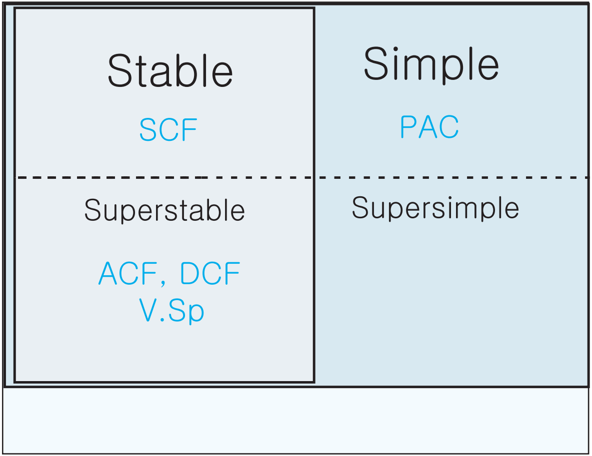
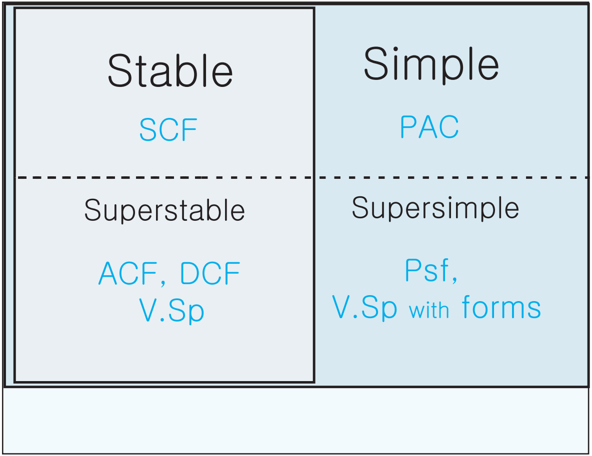
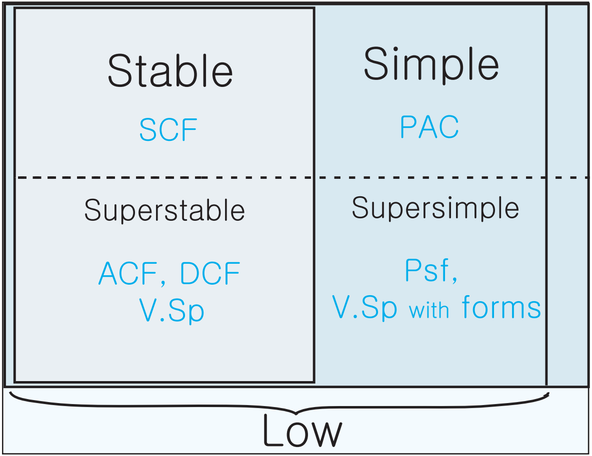

Stable
SCF
Superstable
Simple
PAC
Supersimple


Stable
SCF
Simple
PAC
Superstable
ACF, DCF
V.Sp
Supersimple


Stable
SCF
Simple
PAC
Superstable
ACF, DCF
V.Sp
Supersimple
Psf,
V.Sp with forms


Stable
SCF
Simple
PAC
Superstable
ACF, DCF
V.Sp
Supersimple
Psf,
V.Sp with forms
Low

Definition

- T has *elimination of imaginaries* if for any definable equivalence class a/E, there is finite $b \in \mathcal{M}$ such that they are interdefinable (i.e. for any automorphism fixes the class a/E setwise iff it fixes the tuple b).
- T has *elimination of hyperimaginaries* if for any type-definable equivalence class $\bar{a}/\mathbf{E}$ ($\bar{a}$ possibly infinite), there are definable equivalence classes $\{a_i/E_i\}_i$ such that they are interdefinable.

From now on, we assume T has elimination of imaginaries, and elimination of hyperimaginaries, (so assume $\mathrm{Ltp} = \mathrm{stp}$).
Unless mentioned otherwise T will be simple.

Theorem

(Buechler, Pillay, Wagner) If T is supersimple, then T has elimination of hyperimaginaries.

Fact

If type amalgamation holds for $\mathrm{tp}(a/A)$, *then indeed* $\mathrm{tp}(a/A) = \mathrm{Ltp}(a/A) = \mathrm{tp}(a/\mathrm{acl}(A))$. *Hence type amalgamation holds for any* acl-set *(e.g. a model). Such a type we call an* amalgamation base.

Theorem

(Hart, K, Pillay) Any amalgamation base has a canonical base. Namely, for any amalgamation base $p = \mathrm{tp}(c/A)$ with $A = \mathrm{acl}(A)$, there is smallest $A_0 \subseteq A$ such that $c \perp_{A_0} A$ and $\mathrm{tp}(c/A_0)$ is an amalgamation base. We write $A_0 := \mathrm{Cb}(c/A)$.

Observation

Let $B_i = \mathrm{acl}(B_i) \subseteq B$ $(i = 0, 1)$. If $A \perp_{B_0} B$ and $A \perp_{B_1} B$, then $A \perp_{B_0 \cap B_1} B$: Note that $C := \mathrm{Cb}(A/B) \subseteq B_i$, $\therefore$ by transitivity, $A \perp_C B_0 \cap B_1$ and $A \perp_{B_0 \cap B_1} B$ hold.

Definition

Let $p(x) \in S(A)$ be an amalgamation base. We call $\varphi(x, y)$, *p-stable* if the φ-type $p\lceil\varphi$ has a unique nonforking φ-type extension over any $B \supseteq A$.

Fact

- *$\varphi(x, y)$ is stable iff $\varphi(x, y)$ is p-stable for any amalgamation base $p(x)$.*
- *If $\varphi(x, y)$ is p-stable, then there is $\psi_\varphi(y)$ over $\mathrm{Cb}(p)$, called the φ-definition of p, such that $\models \psi_\varphi(b)$ holds iff $\varphi(x, b)$ is in some nonforking extension of p.*

Theorem

(K) Assume T is supersimple, and $p(x)$ is an amalgamation base. Then
$\mathrm{Cb}(p) = \{$*names of* $\psi_\varphi(y) |\ \varphi(x, y)$ *is* p*-stable* $\}$.

Definition

By a *Morley sequence* of $p \in S(A)$, we mean an A-indiscernible and independent sequence $I = \langle a_i |\ i < \omega \rangle$ in p.

$\mathrm{Cb}(p) \subseteq I$ where I is a Morley sequence of p.

Consider a vector space $V = (V, +, r)_{r \in F}$. For acl-subsets A, B of V, we have $A \mathop{\smile\!\!\!\!\!\!|}_{A \cap B} B$. Equivalently, when finite dimensional, $\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$.

Definition

- T is called *modular* if $\smile\!\!\!\!\!\!|$ satisfies above relations between acl-sets.
- T is said to be *1-based* if for any indiscernible sequence $I = \langle a_i |\ i < \omega \rangle$, $I - a_0$ is Morley over a_0.

Observation

The following are equivalent.
(i) T is 1-based.
(ii) For any $p = \mathrm{tp}(c_0/A)$, $\mathrm{Cb}(c_0/A) \subseteq \mathrm{acl}(c_0)$.
(iii) T is modular.

Proof: (i)$\Rightarrow$(ii) Let $e := \mathrm{Cb}(c_0/A)$. Now there is a Morley sequence $J = \langle c_i | i \leq \omega \rangle$ of p. Then $c_\omega \mathop{\smile\!\!\!\!\!\!|}_A I$ where $I = J - \{c_\omega\}$. Since $c_\omega \mathop{\smile\!\!\!\!\!\!|}_e AI$ and $e \subseteq I$, we have $c_\omega \mathop{\smile\!\!\!\!\!\!|}_I A$. Hence $e = \mathrm{Cb}(c_\omega/A) = \mathrm{Cb}(c_\omega/I)$. By 1-basedness, $c_\omega \mathop{\smile\!\!\!\!\!\!|}_{c_0} I$. Hence $e \in \mathrm{acl}(c_0)$.

(ii)$\Rightarrow$(iii) Given A, B, we have $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{Cb}(A/B)} B$ and $\mathrm{Cb}(A/B) \subseteq \mathrm{acl}(A)$. Hence by transitivity, $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(A) \cap \mathrm{acl}(B)} B$.

(iii)$\Rightarrow$(i) Indiscernible $I = \langle c_i |\ i < \omega \rangle$ is given. (iii) implies $c_0 \downfree_A c_1 c_2 ...$ where $A = \mathrm{acl}(c_0) \cap \mathrm{acl}(c_1 c_2 ...)$. Now, $tp(c_1 c_2 .../\mathrm{acl}(c_0)) = tp(c_2 c_3 .../acl(c_0))$. Hence $A = \mathrm{acl}(c_0) \cap \mathrm{acl}(c_2 c_3 ...)$. Again, $tp(c_0/\mathrm{acl}(c_2 c_3 ...)) = tp(c_1/\mathrm{acl}(c_2 c_3 ...))$, and thus $A = \mathrm{acl}(c_1) \cap \mathrm{acl}(c_2 c_3 ..)$. Then, due to (iii) again, $c_1 \downfree_A c_2 c_3 ...$. From transitivity, we have $c_1 \downfree_{c_0 A} c_2 c_3 ...$. Then $c_1 \downfree_{c_0} c_2 c_3 ...$ since $A \subseteq \mathrm{acl}(c_0)$. Thus (i) follows.